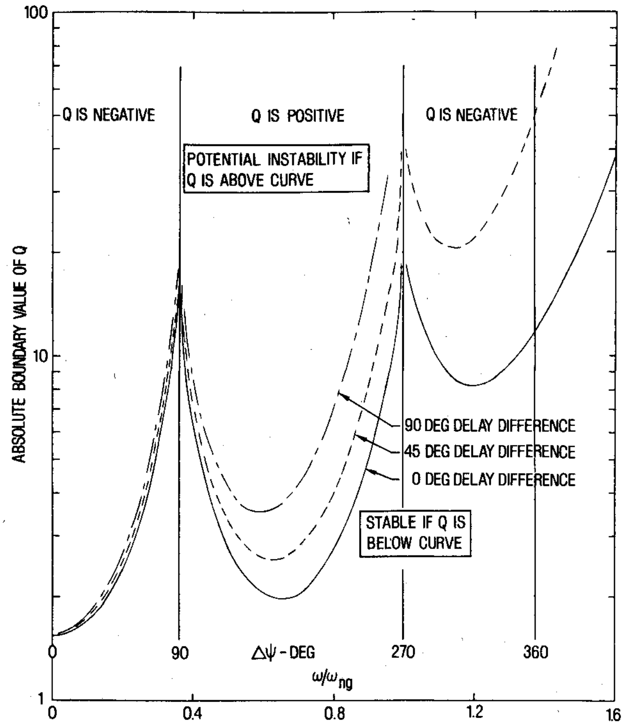

Fig. 3 Double-firing boundary chart.

a and *b*. If the value of Q for the system has the same algebraic sign as the boundary value of Q in that region, and if the absolute value of Q is greater than the boundary value plotted in Fig. 3, then a double-firing instability will occur if the worst-case disturbance torque is applied. If the absolute value of Q is less than the boundary value, then no double-firing instability can occur.

For a given structural damping and bending frequency, $\Delta\psi$ will increase with increases in rate gyro damping, filter time constant, or valve delays, or with a decrease in rate gyro natural frequency. A decrease in $\Delta\psi$ occurs for the reverse trends. Instability in the left-hand negative Q region, therefore, will tend to require a high-frequency lightly damped rate gyro with short filter time constant and valve delays. Instability in this region may be of practical concern for some systems, but for the systems studied by the author the left-hand region is of no interest because the values of Q are usually positive and $\Delta\psi$ is always greater than 90 deg. In the right-hand negative Q region, instability requires relatively low rate gyro frequencies and long filter and valve delays; however, the values of Q required for instability in the right-hand region are so large that instability in this region is probably not a matter of practical concern for any real system. The middle positive Q region is, however, of practical interest for the systems studied by the author. It is seen that the difference between on and off delays affects the curves significantly, and that the use of average delay gives an overly conservative boundary value (Q smaller than necessary) if the off delay is less than the on delay, as it usually is for real solenoid valves.

Acknowledgments

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References

[1] Hecht, C. and Loesch, F.C., "Work/Energy Analysis of Bending Limit Cycles in a Deadband Attitude Control System," *Journal of Spacecraft and Rockets*, Vol. 13, July 1976, pp. 406-415.

Method for Error Budgeting of Inertial Navigation Systems

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Nomenclature

A,B	= general matrices of the same dimensions
C	= covariance matrix of a variate (denoted by subscript)
E	= expected-value operator
F	= $n \times n$ matrix as a function of time
G	= $n \times l$ matrix as a function of time
I	= identity matrix
i,j,k	= integers
l	= dimensions of the forcing noise vector u
m	= number of missions in rms performance estimates
n	= dimension of the modeled error state vector x
P	= matrix projecting x onto r or v (denoted by subscript)
Q	= lower-triangular matrix such that $Q^TQ = W$
R	= upper-triangular matrix such that $R^TR = C$
r	= position error vector
S	= upper triangular matrix = RQ^T
s,t	= time, $0 \leq s \leq t \leq T$
T	= total time over which system performance is specified
$(\ )^T$	= transpose of a matrix or vector
trace	= sum of diagonal elements of a square matrix
u	= l-dimensional uncorrelated noise vector
v	= velocity error vector
W	= symmetrical weighting matrix
x	= n-dimensional state vector of the system error model
Φ	= $n \times n$ state transition matrix
ρ	= rate of increase of rms position error
0	= a submatrix of zeros

Introduction

ERROR covariance analysis has been used as a tool for determining the design requirements of inertial sensor systems.[1] It is a way of estimating overall system performance as a function of the performance of subsystems and components which contribute to system errors. Error budgeting requires solving the inverse problem, which is that of finding an allocation of allowable errors (error budget) among the subsystems such that the system meets a specified set of performance criteria. For inertial navigation and guidance systems, these performance requirements are usually specified in terms of such statistical quantities as the rms position or velocity errors, or the time-rate-of-growth of rms position errors.[2] A conventional approach for determining such statistics for a particular error budget is to integrate the matrix differential equations defining error covariance dynamics for the modelled system and mission, using the error budget to define the initial covariances and covariance dynamics. In order to define an error budget which meets the specified requirements, this process is repeated, using engineering judgement to adjust individual budget allocations

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until a satisfactory error budget is obtained. Unfortunately, this process tends to be costly in computer time.

An approach has been found in which the system performance statistics can be represented by the products of pairs of matrices, one of which contains the error budget and the other of which (a "weighting matrix") characterizes the dynamics of the systems applications. Using this approach, each iteration of the error budget requires in the order of n^2 multiplies, where n is the number of "states" in the system error model. The weighting matrix can also characterize rms system performance for an ensemble of mission scenarios, which is particularly appropriate for tactical systems.

Error Model

The dynamics of navigational errors can be described by a linear forced system

$$\dot{x}(t) = F(t)x(t) + G(t)u(t) \tag{1}$$

The components of the vector x include position and velocity errors; the matrices F and G describe the mission time-history and its effect on error dynamics; and the components of u describe time-random error sources. The covariance of x can be represented at any time t by the following integral equation[3]:

$$C_x(t) = \Phi(t,0)C_x(0)\Phi^{\mathrm{T}}(t,0) + \int_0^t \Phi(t,s)G(s)C_u G^{\mathrm{T}}(s)\Phi^{\mathrm{T}}(t,s)\mathrm{d}s \tag{2}$$

where C_u is the covariance of u and $\Phi(t,s)$ is a solution to the differential equation

$$\dot{\Phi}(t,s) = F(t)\Phi(t,s) \tag{3}$$

such that

$$\Phi(s,s) = I \tag{4}$$

Φ is called a "state transition matrix." The initial value $C_x(0)$ and the constant matrix C_u are specified by the error budget.

Mean-Squared Position Error at Time t

For some integer i_r, the position errors are components i_r through i_r+2 of the state vector x. Then the position error can be represented by a formula of the type

$$r = P_r x \tag{5}$$

where the matrix P_r is of the sort

$$P_r [0 \vdots I \vdots 0] \tag{6}$$

In particular, P_r contains zeros except in columns i_r through i_r+2. Then the mean-squared position error can be represented as a matrix trace:

$$\begin{aligned} E[|r|^2] &= E[\mathrm{trace}(rr^{\mathrm{T}})] \\ &= E[\mathrm{trace}(P_r x x^{\mathrm{T}} P_r^{\mathrm{T}})] \\ &= \mathrm{trace}[C_x(P_r^{\mathrm{T}}P_r)] \end{aligned} \tag{7}$$

(The trace of a square matrix is the sum of its diagonal terms.) In Eq. (7) we have used the fact that, for any two matrices A and B of the same dimensions

$$\mathrm{trace}(AB^{\mathrm{T}}) = \mathrm{trace}(B^{\mathrm{T}}A) \tag{8}$$

This fact can be exploited further by using Eq. (2) for C_x, with the following result:

$$\begin{aligned} E(|r(t)|^2) &= \mathrm{trace}(C_x(t)P_r^{\mathrm{T}}P_r) \\ &= \mathrm{trace}\left\{\left[\Phi(t,0)C_x(0)\Phi^{\mathrm{T}}(t,0) + \int_0^t \Phi(t,s)G(s)C_u G^{\mathrm{T}}(s)\Phi^{\mathrm{T}}(t,s)\mathrm{d}s\right][P_r^{\mathrm{T}}P_r]\right\} \\ &= \mathrm{trace}[C_x(0)W_x(t,F,P_r)] \\ &\quad + \mathrm{trace}[C_u W_u(t,F,G,P_r)] \end{aligned} \tag{9}$$

where the weighting matrices

$$W_x(t,F,P_r) = \Phi^{\mathrm{T}}(t,0)P_r^{\mathrm{T}}P_r\Phi(t,0) \tag{10}$$

$$W_u(t,F,G,P_r) = \int_0^t G^{\mathrm{T}}(s)\Phi^{\mathrm{T}}(t,s)P_r^{\mathrm{T}}P_r\Phi(t,s)G(s)\mathrm{d}s \tag{11}$$

In Eq. (9), the mean-squared position error is represented by the sum of the traces of two matrix products. In each product, one matrix is specified by the error budget, and the other matrix somehow characterizes the sensitivity of the performance statistics to the error budget. Note that computation of the trace of a product of two $n \times n$ matrices requires only n^2 multiplies and adds, whereas the computation of the matrix product would require n^3 multiplies and adds. The weighting matrix $W_u(t,F,G,P_r)$ will be $l \times l$, where l is the dimension of u. Therefore, computation of the mean-squared position error requires $n^2 + l^2$ multiplies and adds.

Rms Position Error Rate

One could use the values of the weighting matrices as functions of time to compute the rms position error as a function of time, and then estimate the rms position error rate by fitting a straight line to the resulting function. However, one can also define an equivalent weighting matrix to obtain this statistic directly.

The least-squares estimate of the rms position error rate ρ such that

$$|r(t)|^2 \approx \rho^2 t^2 \qquad (0 \le t \le T) \tag{12}$$

is

$$\rho^2 = \frac{5}{T^5}\int_0^T |r(t)|^2 t^2 \mathrm{d}t \tag{13}$$

$$= \mathrm{trace}[C_x(0)W_{x2}(F,P_r)] + \mathrm{trace}[C_u W_{u2}(F,G,P_r)] \tag{14}$$

where the weighting matrices are defined by the following formulas:

$$W_{xk}(F,P_r) = (2k+1)T^{-2k-1}\int_0^T t^k W_x(t,F,P_r)\mathrm{d}t \tag{15}$$

$$W_{uk}(F,G,P_r) = (2k+1)T^{-2k-1}\int_0^T t^k W_u(t,F,G,P_r)\mathrm{d}t \tag{16}$$

for $k=2$. (One can derive similar formulas with $k=1$ to give weighting matrices for random walk rates.)

Time-rms Velocity Errors

Velocity errors are also components of the state vector x. Therefore, they can also be represented by a formula of the type

$$v = P_v x \tag{17}$$

where the matrix P_v has the form shown in Eq. (6). As a result, the formulas for position errors can also be applied to velocity errors, but with P_v replacing P_r. Time-rms velocity error is defined as

$$|v|_{\text{time-rms}} = \left\{ \frac{1}{T} \int_0^T E[|v(t)|^2] \mathrm{d}t \right\}^{1/2} \tag{18}$$

As a result, time-rms velocity error can be expressed in terms of weighting matrices

$$|v|^2_{\text{time-rms}} = \text{trace}[C_x(0) W_{x0}(F,P_v)] + \text{trace}[C_u W_{u0}(F,G,P_v)] \tag{19}$$

where the weighting matrices are given by Eqs. (15) and (16), but with P_r replaced by P_v, and $k=0$.

Weighting Matrices for Multimission Applications

The matrix functions F and G are mission-dependent. One may prefer to compute rms statistics over an ensemble of missions, with associated functions F_1, G_1; F_2, G_2; $\cdots$; F_m, G_m. The equivalent weighting matrices for this case are

$$W_{xk}(P) = \frac{1}{m} \sum_{i=1}^{m} W_{xk}(F_i,P) \tag{20}$$

$$W_{uk}(P) = \frac{1}{m} \sum_{i=1}^{m} W_{uk}(F_i,G_i,P) \tag{21}$$

for $P=P_r$ or P_v.

Square-Root Implementation

If R is the upper-triangular square root of C and Q is the lower-triangular square root of W,† then

$$\text{trace}(CW) = \sum S_{ij}^2 \tag{22}$$

where

$$S = RQ^{\mathrm{T}} \tag{23}$$

For $n \times n$ matrices, this computation requires in the order of $n^3/6$ multiplies, whereas computing trace (CW) requires only n^2 multiplies. However, if C is a diagonal matrix (i.e., the error budget includes no nonzero correlations), then it requires in the order of n^2 multiplies. This approach may be preferable in the case that the error budget is given in terms of rms values, because it avoids having to square them to obtain the equivalent covariance matrices.

Limitations

The model in Eq. (1) does not apply to "aided" inertial navigation, in which other navigational sensors are used for estimating and correcting the error states. It is not known whether an equivalent weighting matrix can be defined for that application.

References

[1] O'Donnell, C.F., *Inertial Navigation Analysis and Design,* McGraw-Hill, New York, 1964.

[2] Tech. Exhibit ENAC-77-1, "Characteristic for a Moderate Accuracy Inertial Navigation System (INS)," Aeronaut. Sys. Div., Air Force Systems Command, WPAFB, Ohio, March 1978.

[3] Bucy, R.S. and Joseph, P.D., *Filtering for Stochastic Processes with Applications to Guidance,* Interscience, New York, 1968, pp. 24-25.

†W may be only positive semidefinite, rather than positive definite. It is still possible to obtain a square root of W, however.

Bounds on the Solution to a Universal Kepler's Equation

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Introduction

A FUNDAMENTAL problem in two-body orbital mechanics is the solution of Kepler's equation, which describes the initial-value problem relating time and position in a known orbit from specified initial position r_0 and velocity v_0 at an arbitrary epoch t_0. In this Note upper and lower bounds on the position in orbit at a specified value of time are derived for a class of universal formulations of Kepler's equation valid for all conic orbits and for the classical formulations for elliptic and hyperbolic orbits. Bounds on the solution provide efficient starting values for iterative numerical algorithms and eliminate unnecessary computations outside of the solution bounds. This is especially useful in the case of onboard computation.

Several numerical examples illustrating the use of these bounds in example problems from the textbook literature are given in Ref. 1 which is a generalization of preliminary results reported in Ref. 2. These examples demonstrate that a starting value for the iteration which is close to the desired solution is obtained using the bounds.

Analysis

Consider formulations of Kepler's equation in which a generalized position variable u, which describes the location of a body in orbit relative to its initial position at an arbitrary epoch t_0, satisfies a Sundman transformation[3] of the form

$$\mathrm{d}u/\mathrm{d}t = K/r \qquad u(t_0) = 0 \tag{1}$$

where K is a constant and r is the magnitude of the radius vector. The position variable u can then be interpreted as a regularized time variable.[4] Formulations in which the Sundman transformation is satisfied include 1) the classical Kepler's equation for an elliptical orbit in which u is the difference in eccentric anomaly $E-E_0$ and $K=\sqrt{\mu/a}$; 2) the classical hyperbolic Kepler's equation in which u is the difference in hyperbolic eccentric anomaly $H-H_0$ and $K=v_\infty$(hyperbolic excess speed); 3) Barker's equation for a parabolic orbit; and 4) various universal formulations,[5-9] including that of Battin,[5] in which $u=x$ and $K=\sqrt{\mu}$ independent of the orbit.

Since the value of the radius r depends on the position in the orbit, Eq. (1) can be written as

$$r(u)\mathrm{d}u = K\mathrm{d}t \qquad u(t_0) = 0 \tag{2}$$

Integrating, one obtains

$$I(u) \triangleq \int_0^u r(\xi)\mathrm{d}\xi = K(t-t_0) \tag{3}$$

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